REVISITING THE INVERTED PENDULUM ON A MOVING CART

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Abstract
Many control engineers found the inverted pendulum on a moving cart very interesting and intriguing because many control principles can be shown. Two examples are to balance a stick on the end of one's finger and the other one is to balance a rocket during liftoff. This is a highly non-linear system but for simplicity purposes the overall system is linearized making some assumptions. Therefore, in this paper linear first-order differential equations will be utilized for modeling the system along with the MATLAB (MATrixLABoratory) software for simulation purposes. Finally, the same MATLAB simulations will demonstrate stabilization of such system using its ACKER and LSIM built-in functions. The vigorous mathematical theory of this project is presented using the Lagrange's method along with some basic knowledge in Linear Algebra and Differential Equations due to simulations.

Introduction
This work discusses a mathematical model of an inverted pendulum on a moving cart, a common toy control system. Many control engineers found this system interesting and intriguing because many control principles can be shown. Note that this is the same problem as balancing a stick on the end of one’s finger. At first glance this might not be exciting; however, one may note that it is also related to the problem of balancing a rocket during liftoff which is a more exciting problem to tackle.

The same issues are discussed, as in the biological systems, that is, modeling, mathematical analysis and stability analysis. The resulting model is a non-linear and highly unstable system. It is also a notoriously difficult system for which to design a good controller. Thus, there were a lot of efforts in finding various methods to stabilize such a difficult system. One simple method that stabilizes this system by approximating the non-linear model with the linear model is presented in this report.

A description of the system is given followed by a brief introduction of the Lagrange’s method [1] for modeling the system. The mathematical analysis (finding steady states, linearization, etc.) is also presented. Finally, the pole placement using state-variable feedback method is introduced along with Ackerman’s formula to stabilize the pendulum. Moreover, the performance of the design is tested by simulating the system using the computer software MATLAB [2]. The reader may obtain a more theoretical perspective from references [3] and [4]. A very brief reference can also be found in [5].

Description
The setup of this system is shown in the following figure [1]:
The system consists of a cart of mass $M$ that slides in one dimension $y$ on a horizontal surface, with a ball of mass $m$ at the end of a rigid massless pendulum of length $l$. The cart and the ball are treated as point masses, with the pivot at the center of the cart. There is assumed to be no friction and no air resistance. It is shown as input a horizontal force $f$. The other signals shown are the angle $\theta$ and the position of the ball with coordinates $(y_2, z_2)$ which are $y_2 = y + l\sin(\theta)$ and $z_2 = l\cos(\theta)$.

**Modeling**

The derivation of the equations governing the motion of this system is based on the method developed by the eighteenth-century French mathematician Lagrange. The differential equations that result from this method are known as Lagrange’s equations derived from Newton’s laws of motion.

The fundamental principle of Lagrange’s equations is the representation of the system by a set of generalized coordinates $q_i (i = 1, 2, 3, \ldots, r)$, one for each independent degree of freedom of the system, completely incorporating the constraints unique to the system. After having defined the generalized coordinates, the kinetic energy $T$ is expressed in terms of these coordinates and their derivatives and the potential energy $V$ is expressed in terms of the generalized coordinates. Next the Lagrangian function $L = T(q_1, q_2, \ldots, q_r, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_r) - V(q_1, q_2, \ldots, q_r)$ is formed. And finally the desired equations of motion are defined using the Lagrange’s equations

$$\sum_{i=1}^{r} \frac{\partial}{\partial \dot{q}_i} \left( \frac{\partial L}{\partial q_i} \right) - \frac{\partial L}{\partial q_i} = Q_i$$

for $i = 1, 2, 3, \ldots, r$ where $Q_i$ denotes generalized forces (forces and torques) that are external to the system. Each of the differential equations in the set will be a second-order differential equation, so a dynamic system with $r$ degrees of freedom is represented by $r$ second-order differential equations. If one state variable is assigned to each generalized coordinate and another to the corresponding derivative, we end up with $2r$ equations. So, a system with $r$ degrees of freedom is of order $2r$.

A typical application of Lagrange’s equations is to define the motion of a collection of bodies that are connected together in some manner such as our system. The motion of the system is uniquely defined by the displacement of the cart from some reference point and the angle that the pendulum rod makes with respect to the vertical axis. So, it can be concluded that the system has only two degrees of freedom, and that the dynamics must be expressed in terms of the corresponding generalized coordinates chosen ($y, \theta$). The details of the derivation of the exact equations of motion are given in [3]. The equations are $$(M + m)\ddot{y} + m\cos(\theta)\ddot{\theta} - ml(\dot{\theta})^2 \sin(\theta) = f$$ and $m\cos(\theta)\ddot{y} + ml^2 \dddot{\theta} - mg\sin(\theta) = 0$ where $g = 9.81$ m/sec$^2$, the acceleration due to gravity.
Analysis
The two previous equations are nonlinear owing to the presence of the trigonometric terms \( \sin(\theta) \) and \( \cos(\theta) \) and the quadratic terms \( (\dot{\theta})^2 \) and \( \ddot{\theta} \). If the pendulum is stabilized; however, then \( \theta \) is kept small. This justifies the approximation \( \cos(\theta) \approx 1 \) and \( \sin(\theta) \approx 0 \). It can also be assumed that \( \dot{\theta} \) and \( \ddot{\theta} \) are kept small so that the quadratic terms are negligible. Another way to observe this is to linearize the above equations about an equilibrium position, of which there are two: \( (y, \theta) = (0, 0) \) and \( (y, \theta) = (0, \pi) \); that is, the pendulum is either up or down. The first case is considered because is more interesting. Thus, the linearized model is as follows: \((M + m) \ddot{y} + ml \dddot{\theta} = f\) and
\[ml\dot{y} + ml^2 \ddot{\theta} - mgl\theta = 0\]. Defining the state vector \( x = [y, \theta, \dot{y}, \dot{\theta}]^T \) where \( T \) denotes transpose, the state-space representation (standard matrix form) of the linearized system can be found. The state-variables are the cart position, rod angle, cart velocity and rod angular velocity. The linearized model consists of four first-order differential equations as follows:
\[\frac{dy}{dt} = \dot{\theta}, \frac{d\theta}{dt} = \ddot{\theta}, \frac{d\dot{y}}{dt} = \ddot{\theta}, \frac{d\ddot{\theta}}{dt} = \dddot{\theta}\] and \( u = f = \) external force. Having converted the system to the state-space representation, it is ready for stability analysis as well as for performance analysis by simulation. Certain numerical values are selected which are: \( M = 5 \) kg (mass of cart), \( m = 0.5 \) kg (mass of pendulum), \( l = 1 \) m (length of pendulum, and \( g = 9.81 \) m/sec\(^2 \) (acceleration due to gravity).

Stability Analysis
With the numerical values the following system is obtained:
\[\dot{x} = A x + B u\]
where \( A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{mg}{M} & 0 & 0 \\ 0 & \frac{(M+m)g}{ml} & 0 & 0 \end{bmatrix}\) and \( B = \begin{bmatrix} 0 \\ 1 \\ -1 \\ -1 \end{bmatrix}\) and \( u = f = \) external force. Having converted the system to the state-space representation, it is ready for stability analysis as well as for performance analysis by simulation. Certain numerical values are selected which are: \( M = 5 \) kg (mass of cart), \( m = 0.5 \) kg (mass of pendulum), \( l = 1 \) m (length of pendulum, and \( g = 9.81 \) m/sec\(^2 \) (acceleration due to gravity).

To gain some insight on the system, its open-loop poles are determined to be at 0, 0, 3.283, and -3.283. The poles at \( \pm 3.283 \) arise from the angle subsystem and are involved with \( \theta \) and \( \dot{\theta} \). The two poles at zero evolve from the Newton’s system relating to the horizontal position and involve \( y \) and \( \dot{y} \). Now let us use the state feedback law \( u = -Kx \) to stabilize the system where \( K \) is the feedback gain. Using this method, the rod angle should be measured, the rod angular velocity, the cart position and the cart velocity. To accomplish this, one way is to place potentiometers at the rod pivot point for the rod angle and on one wheel for the cart position. To measure the rod angular velocity and the cart velocity, it is necessary to place tachometers at the rod pivot point and on the other wheel. This involves much instrumentation and the apparatus might be complicated to be built. There are other designs such as output feedback control system where only the output need to be measured, namely the rod angle and the cart position. This is beyond the scope of this paper.

Now the purpose is to evaluate \( K \) using Ackermann’s formula given by
\[K = e_n^T U_n^{-1} \Delta^D(A)\] where \( U_n \) is the reachability matrix given by \( U_n = [B \ A B \ \ldots \ A^{n-1}B] \) and \( e_n^T = [0 \ 0 \ \ldots \ 0 \ 1] \) is the last row of the \( n \times n \) identity matrix. \( \Delta^D(A) \) is the desired characteristic polynomial evaluated at \( A \). It is an \( n \times n \) matrix polynomial. Since out system is a fourth order, \( n = 4 \). So, the control law has the form \( u = -Kx = \)
The desired closed-loop poles are selected to be \( s = -1 \pm i \) and \( s = -2 \pm 2i \). This yields the desired characteristic polynomial of \( \Delta^D(s) = s^4 + 6s^3 + 18s^2 + 24s + 16 \) and the 4 x 4 matrix polynomial is given by \( \Delta^D(A) = A^4 + 6A^3 + 18A^2 + 24A + 16I \). The reachability matrix is \( U_4 = [B\ AB\ A^2B\ A^3B] \). The following MATLAB code is written:

```matlab
% here is the inverted pendulum on a cart A matrix
A=[0 0 1 0;0 0 0 1;0 -0.98 0 0;0 10.78 0 0];
% this is the B matrix
B=[0;0;0.2;-0.2];
% eigenvalues of A
b=eig(A);
% the pole placement matrix P is
P=[-1+i;-1-i;-2+2i;-2-2i];
% the gains k1 k2 k3 and k4 are given by
K=acker(A,B,P);
% the closed loop A1 matrix is given by
A1=A-B*K;
% plotting
% initial conditions
x0=[0.1 0.1 0 0];
% matrices to use the lsim function
B1=[0;0;0;0];C1=[0 0 0 0];D1=0;
closedloopsystem=ss(A1,B1,C1,D1);
t=[0:0.01:18];
% input u is zero too
u=0*t;
% here is the lsim function
[y,T,x]=lsim(closedloopsystem,u,t,x0);
plot(T,x(:,1),T,x(:,2),'--')
xlabel('time in seconds')
ylabel('cart position & rod angle theta')
title('Inverted Pendulum Closed Loop Response')
```

**Figure 2. Simulation code for the inverted pendulum on a moving cart**

The MATLAB command `acker` gave the following gain numerical values 
\( K = [-8.1633 -152.0633 -12.2449 -42.2449] \). The initial conditions of the simulation at \( t = 0 \) are rod angle \( \theta(t) = 0.1 \) rad, cart position \( y(t) = 0.1 \) m, rod angular velocity \( \dot{\theta}(t) = 0 \) rad/s and cart velocity \( \dot{y}(t) = 0 \) m/s. The simulation graph is shown in Figure 3.
It can be seen that after about 6 seconds the system comes to rest with the rod balanced at $\theta = 0$ and the cart at the position $y = 0$. The rod response $\theta(t)$ is quite good, and it remains small so that the linear approximation holds. However, the position $y(t)$ increases to $0.43$ m before it returns to zero. This is necessary since the cart must run up under the rod to balance it before coming back to the origin $y = 0$.

Another simulation: Using now initial conditions of $x_0 = [0 \ 0 \ 0.1 \ 0.1]$ the following simulation graph is obtained.

Pole-placement design using full state feedback is sometimes unsuitable, since it does not take advantage of the full design freedom in a problem. The overall performance can be further improved by using output feedback design where the zeros of the transfer function from $u(t)$ to $y(t)$ and from $u(t)$ to $\theta(t)$ are manipulated. In the pole-placement method zeros are not used which is the main drawback of the method. Only poles of the desired characteristic polynomial are used.
Conclusion
The modeling of a very difficult control system was demonstrated using the Lagrange’s method which is an effective method in deriving the equations of motion. By linearizing the system around the origin (0,0) where the pendulum is up, a very good approximation was obtained to the actual nonlinear system. Obtaining the linearized model, a stabilizing controller was designed using pole-placement with state-variable feedback along with the Ackermann’s formula. This method has some drawbacks which can be improved by using a different method. Overall, the modeling of such a difficult system was meaningful and successful as indicated by the simulation results of the linearized approximation. It can be used as a sample for modeling more applicable interesting systems which can help for the advancement of today’s technology.

References: